



Static Pricing of Exotic Derivatives Under Conditional Value-at-Risk (CVaR) in Incomplete Markets

Benyanee Kosapong¹ · Ratinan Boonklurb¹ · Udomsak Rakwongwan²

Accepted: 21 March 2025
© The Author(s) 2025

Abstract

This paper optimizes portfolios using Conditional Value-at-Risk (CVaR) as a risk measure with tradable assets such as cash, call and put options on the S&P 500. It develops a static pricing model for the buy-and-hold strategy and evaluates dynamic strategies. The CVaR model is compared to the mean-variance model, demonstrating its ability to minimize CVaR while achieving required returns and showing sensitivity to parameters, namely, the shape of the variance-gamma distribution, volatility, confidence level, and required return. Although the mean-variance model results in lower standard deviation, the CVaR-based optimization consistently yields lower CVaR values. For portfolios with liabilities, the model incorporates hedging strategies for standard European call option and Exotic options such as quadratic, log, digital, sine, and butterfly spread options in an incomplete market. The CVaR-based approach demonstrates risk reduction and robustness in the backtesting.

Keywords Portfolio optimization · Conditional Value-at-Risk · Mean-Variance · Variance gamma distribution · Exotic options

Mathematics Subject Classification 90C05 · 91G10 · 91G20 · 91G50

✉ Udomsak Rakwongwan
udomsak.ra@ku.th

Benyanee Kosapong
benyaneeda@gmail.com

Ratinan Boonklurb
ratinan.b@chula.ac.th

¹ Department of Mathematics and Computer Science, Faculty of Science, Chulalongkorn University, Bangkok 10330, Thailand

² Department of Mathematics, Faculty of Science, Kasetsart University, Bangkok 10900, Thailand

1 Introduction

The pricing and hedging of financial derivatives are fundamental concepts in finance that ensure market efficiency and stability. Accurate pricing enables fair trade between issuers and buyers while helping them manage risk and prevent significant financial losses. Hedging serves as a form of insurance by allowing investors to take positions in derivatives that offset potential losses in other investments. In complete markets with a unique risk-neutral probability measure, financial derivatives can be priced and hedged with precision. The absence of arbitrage opportunities allows for unique derivative pricing through replication strategies, directly linking each derivative's value to its underlying asset. Moreover, these markets enable the creation of perfect hedges that dynamically offset derivative position risks. Seminal academic works such as Black and Scholes (1973) and Merton (1973) laid the theoretical foundation for derivative pricing in complete markets, while subsequent research by Hull (2019) and Wilmott (2013) expanded on these models and explored the practical implementation of hedging strategies.

Markets are often incomplete due to several constraints. These include limits on asset trading volumes and regulatory barriers. Additional complications arise from transaction costs such as bid-ask spreads and trading commissions, which impede continuous delta hedging strategies. Unlike complete markets, where a unique risk-neutral probability measure enables perfect replication of contingent claims, incomplete markets require alternative approaches to pricing and hedging. Methods such as indifference pricing and static hedging offer practical solutions by incorporating investor preferences and market frictions. In addition, the indifference pricing and hedging, rooted in utility theory, represent crucial methodologies to address the challenges posed by market imperfections. It offers a framework for determining the price at which an investor is indifferent between buying or selling a derivative or holding a current position, taking into account their risk aversion. This approach not only accommodates the intrinsic limitations of the market, but also reflects individual preferences, such as probabilistic views of the market value, risk preferences, and current positions, leading to more personalized hedging strategies that aim to minimize risk while accounting for the inevitable trade-offs in such constrained environments; see, e.g., Pennanen (2012, 2014).

Unlike other areas within Financial Mathematics that predominantly rely on stochastic analysis, the realm of pricing and hedging in incomplete markets primarily revolves around optimization theory, with convex analysis being the fundamental technique. Classic references on indifference pricing are Carmona (2008) and Hodges (1989). In addressing these issues, many studies in incomplete markets utilize duality theory, deriving no-arbitrage bounds; see, e.g., Guo and Oblój (2019) and Ilhan et al. (2004, 2009). However, research focusing on computational methodologies remains scarce. Armstrong et al. (2018) established a framework for indifference pricing under an exponential utility function and numerically determined indifference prices and hedging strategies for exotic

index options. Yamphram et al. (2023) solved the same problem but under the mean-variance criteria, initially presented in Markowitz's Nobel Prize winning work, Markowitz (1968). Portfolio optimization under Value-at-Risk (VaR) and Conditional Value-at-Risk (CVaR) has received significant scholarly attention, as evidenced by works such as Gaivoronski and Pflug (2005), Rockafellar and Uryasev (2000, 2002) and Kosapong et al. (2022). In particular, CVaR has garnered more focus as VaR lacks sub-additivity and convexity properties, which prevents it from qualifying as a coherent risk measure Artzner (1997), Artzner et al. (1999), Rockafellar (2015) and Sarykalin et al. (2008). However, a static practical framework for the pricing and hedging of indifference of exotic derivatives using CVaR remains unproposed. Recent CVaR portfolio studies, including Abdul Razak et al. (2019) on traditional assets and Strub et al. (2019) on time-consistent models, overlook critical challenges in incomplete markets for exotic derivatives and transaction costs from dynamic hedging. While Wang et al. (2024) integrate investor preferences for generic assets, and Guo et al. (2019) address background risks, none tackle static hedging for derivatives under real-world frictions.

This study extends the work of Rockafellar and Uryasev (2002) to introduce a comprehensive and practical framework for pricing and hedging derivatives under CVaR, with several key innovations. First, in their foundational work, Rockafellar and Uryasev demonstrated how CVaR can be used for portfolio optimization to minimize potential losses. We extend this by applying their CVaR optimization framework to option hedging through an option-hedge-option approach. Specifically, we determine the price of a given liability by finding the minimum cash the agent would need to maintain their current level of disutility, with the rigorous definition provided in Sect. 2.2. Traditional option hedging in the Black-Scholes framework relies on continuous trading of the underlying asset. While prior works like Derman et al. (2000) and Carr and Wu (2002) introduced static option-based hedging focusing on exact payoff replication, our work takes a different angle by combining the option-hedge-option approach with CVaR optimization. This static setting uses only a buy-and-hold strategy with standard options to hedge exotic options, reducing transaction costs and minimizing dependence on future price movements. Second, we demonstrate our framework through applications in the S&P 500 options market by both standard European call options and exotic derivatives including quadratic, log, digital, sine, and butterfly spread options. Following the findings of Yamphram et al. (2023), this framework allows quantification of dependencies within optimized portfolios, indifference prices, and agent-specific parameters. Finally, we demonstrate its effectiveness in managing tail risk through comparative analysis with mean-variance optimization and empirical validation using historical market data. Our results show that the framework consistently achieves target returns while maintaining robust protection against extreme market movements, offering significant practical value for risk management in institutional investment.

The paper unfolds as follows. Section 2 introduces the pricing and hedging framework, mathematically formulated as a portfolio optimization model. Most of the definitions, including those of VaR, CVaR, indifference pricing for selling and buying, are mathematically and rigorously defined in this section. Section 3 describes the market

in which we illustrate the application of the framework and introduces a probabilistic model used to determine the view on the future prices of considered assets. Section 4 shows the optimized portfolios, indifference prices for buying and selling, and hedging strategies. The efficiency of the hedging portfolios determined by the hedging residuals and their comparison to those under the mean-variance criteria is demonstrated here. Finally, Sect. 5 concludes the study of our paper.

2 Mathematical Models

Unlike pricing in the Black and Scholes framework in which stochastic calculus plays an important role, indifference pricing is based on portfolio optimization and convex analysis. In this section, we introduce the indifference pricing model for exotic index options when CVaR is viewed as a risk measure.

2.1 Optimization Model with a Given Contingent Claim Under CVaR

VaR is a widely used risk management tool that quantifies the potential loss in value of a financial asset or portfolio over a specified time period at a given confidence level. At its core, VaR answers the question: “What is the maximum amount that we could lose over a certain period at a certain confidence level?” CVaR extends beyond VaR as an advanced risk management metric by measuring both the potential loss at a designated confidence level and the average loss exceeding the VaR threshold. Specifically, CVaR answers the question: “What is the expected loss beyond the VaR threshold over a certain period at a certain confidence level?”

We extend the method proposed by Rockafellar and Uryasev (2000) to incorporate a future contingent claim. Since we focus on an options market written on a single underlying asset, our only stochastic variable is the underlying’s value at maturity time T , denoted as S_T . For a more rigorous model description, we define $(\Omega, \mathcal{F}, \mathcal{P})$ as the probability space describing S_T , where $C_T \in L^0(\Omega, \mathcal{F}, \mathcal{P})$ represents a future liability. Let $f(\mathbf{x}, S_T, C_T)$ denote the loss function associated with the portfolio vector $\mathbf{x} \subseteq \mathbb{R}^n$, where n is the number of trading assets. Then, the probability that the loss $f(\mathbf{x}, S_T, C_T)$ does not exceed $\alpha \in \mathbb{R}$ is given by

$$\Psi(\mathbf{x}, \alpha) = \mathbb{P}(f(\mathbf{x}, S_T, C_T) \leq \alpha).$$

The β -VaR and β -CVaR values for the loss function associated with the portfolio vector \mathbf{x} and the confidence level $\beta \in (0, 1)$, denoted by $\alpha_\beta(\mathbf{x})$ and $\phi_\beta(\mathbf{x})$, can be expressed as,

$$\alpha_\beta(\mathbf{x}) = \min \{ \alpha \in \mathbb{R} \mid \Psi(\mathbf{x}, \alpha) \geq \beta \} \quad (1)$$

and

$$\phi_\beta(\mathbf{x}) = (1 - \beta)^{-1} \mathbb{E} \left[f(\mathbf{x}, S_T, C_T) \mathbb{1}_{\{f(\mathbf{x}, S_T, C_T) \geq \alpha_\beta(\mathbf{x})\}} \right] \quad (2)$$

where the indicator function is defined as

$$\mathbb{1}_{\{f(\mathbf{x}, S_T, C_T) \geq \alpha_\beta(\mathbf{x})\}} = \begin{cases} 1 & \text{if } f(\mathbf{x}, S_T, C_T) \geq \alpha_\beta(\mathbf{x}), \\ 0 & \text{if } f(\mathbf{x}, S_T, C_T) < \alpha_\beta(\mathbf{x}). \end{cases}$$

Optimizing CVaR over all feasible portfolio vectors \mathbf{x} is not trivial, as the mathematical expression of CVaR, as in (2), is complicated. Rockafellar and Uryasev (2000) showed that minimizing CVaR over all feasible portfolios is equivalent to minimizing a newly formulated function $F_\beta(\mathbf{x}, \alpha)$ over $(\mathbf{x}, \alpha) \in \mathbb{R}^n \times \mathbb{R}$. In other words,

$$\min_{\mathbf{x} \in X} \phi_\beta(\mathbf{x}) = \min_{(\mathbf{x}, \alpha) \in X \times \mathbb{R}} F_\beta(\mathbf{x}, \alpha), \tag{3}$$

where

$$F_\beta(\mathbf{x}, \alpha) = \alpha + (1 - \beta)^{-1} \int_{S_T > 0} [f(\mathbf{x}, S_T, C_T) - \alpha]^+ p(S_T) dS_T, \tag{4}$$

and

$$[f(\mathbf{x}, S_T, C_T) - \alpha]^+ = \begin{cases} f(\mathbf{x}, S_T, C_T) - \alpha, & f(\mathbf{x}, S_T, C_T) - \alpha > 0, \\ 0, & f(\mathbf{x}, S_T, C_T) - \alpha \leq 0. \end{cases}$$

Let $C_T, R_i(S_T) \in L^0(\Omega, \mathcal{F}, \mathcal{P})$ be a random claim and return of the i th asset, respectively. We have that the loss (the negative of the return) associating with a portfolio vector \mathbf{x} can be written as

$$f(\mathbf{x}, S_T, C_T) = \frac{C_T}{W} - [x_1 R_1(S_T) + \dots + x_n R_n(S_T)] = \frac{C_T}{W} - \mathbf{x}^T \mathbf{R}(S_T), \tag{5}$$

where $W \in \mathbb{R}$ is an initial wealth.

Note that this framework is flexible with respect to the underlying return distribution. Any well-defined probability density function $p(S_T)$ can be accommodated in this framework, as the return $R(S_T)$ is simply defined as the ratio between payoff at maturity T and initial cost, where S_T is used as a parameter to determine the payoff of each asset.

To utilize traditional techniques to solve the optimization problem for the objective function (4), which becomes particularly challenging to evaluate analytically when dealing with multiple assets and exotic options with nonlinear payoffs, we employ a numerical approximation. As our integral is one dimensional, instead of using a simulation-based method, we approximate the integral using a q -point Gauss-Legendre quadrature rule, as shown in Pennanen and Koivu (2002) and Sullivan (2000). This approach transforms our complex optimization problem into a tractable linear programming problem that can be solved efficiently using standard techniques. The approximated objective function, CVaR, can be written as:

$$\min_{(\mathbf{x}, \alpha) \in X \times \mathbb{R}} \tilde{F}_\beta(\mathbf{x}, \alpha) = \min_{(\mathbf{x}, \alpha) \in X \times \mathbb{R}} \alpha + \frac{1}{(1 - \beta)} \sum_{k=1}^q [C_T/W - \mathbf{x}^T \mathbf{R}(S_T^k) - \alpha]^+ p(S_T^k) w_k, \tag{6}$$

where S_T^k represents the k th root of the Legendre polynomial on the interval $[0, \infty)$, $p(S_T^k)$ is the probability density function of S_T^k and w_k is the corresponding weight associated with S_T^k .

By introducing auxiliary variables $u_k = C_T/W - \mathbf{x}^T \mathbf{R}(S_T^k) - \alpha$ and constraints $u_k > 0$ for $k = 1, 2, 3, \dots, q$, the minimization of CVaR as stated in (3) can be formulated as a linear programming problem as shown below,

$$\min_{(\mathbf{x}, \mathbf{u}, \alpha) \in \mathbf{X} \times \mathbb{R}^q \times \mathbb{R}} \left(\alpha + \frac{1}{(1 - \beta)} \sum_{k=1}^q u_k p(S_T^k) w_k \right), \tag{7}$$

subject to

$$u_k \geq 0, \quad k = 1, 2, 3, \dots, q,$$

$$u_k - C_T/W + \mathbf{x}^T \mathbf{R}(S_T^k) + \alpha \geq 0, \quad k = 1, 2, 3, \dots, q,$$

$$\sum_{i=1}^n x_i = 1,$$

$$\mathbf{x}^T \overline{\mathbf{R}}(S_T) - C_T/W \geq M/W,$$

where $\overline{\mathbf{R}}(S_T)$ is a vector of average returns at the maturity time T , M represents the required capital from the investment and \mathbf{x} is a set of all feasible trading strategies.

2.2 Indifference Prices and Hedging Strategies

Let $\varphi(W, C_T)$ be the minimum CVaR (risk) obtained from the optimization model (7) as a function of the initial wealth W and the future liability C_T . We modeled this

Table 1 Examples of call options in market quotes on May 19, 2020 at 12:00:00

Strike (K)	Bid price	Ask price	Bid size ($\times 100$)	Ask size ($\times 100$)
290	11.51	11.69	96	96
291	10.82	11.00	112	112
292	10.14	10.34	162	112
293	9.49	9.68	162	112
294	8.85	9.04	50	128
295	8.24	8.42	50	128
296	7.65	7.82	50	128
297	7.07	7.25	50	128
298	6.52	6.69	50	128
299	6.00	6.16	50	128
300	5.50	5.65	50	144
301	5.02	5.16	50	144
302	4.57	4.71	50	160
303	4.14	4.28	50	176
304	3.74	3.88	50	192
305	3.37	3.50	100	308

Table 2 Examples of put options in market quotes on May 19, 2020 at 12:00:00

Strike (K)	Bid price	Ask price	Bid size ($\times 100$)	Ask size ($\times 100$)
290	6.72	6.86	144	50
291	6.97	7.18	50	50
292	7.3	7.51	50	50
293	7.7	7.86	128	50
294	7.99	8.23	50	50
295	8.37	8.61	50	50
296	8.77	9.02	50	50
297	9.18	9.44	142	13
298	9.62	9.9	13	13
299	10.09	10.37	125	13
300	10.59	10.87	112	13
301	11.11	11.4	96	109
302	11.65	11.94	80	80
303	12.22	12.52	64	64
304	12.8	13.13	48	48
305	13.46	13.74	32	32

liability as a bounded measurable function within the probability space $(\Omega, \mathcal{F}, \mathcal{P})$, consistent with prior work by Carmona (2008) and Pennanen (2009). Let $\bar{W} \in \mathbb{R}$ and $\bar{C}_T \in L^0(\Omega, \mathcal{F}, \mathcal{P})$ be the initial wealth and initial liability of the agent before selling a derivative with future liability C_T , respectively. The indifference price for selling the claim C_T , see, e.g. Armstrong et al. (2018) and Xu (2006), can be defined as follows:

$$\pi_s(\bar{W}, \bar{C}_T, C_T) = \inf\{W | \varphi(\bar{W} + W, \bar{C}_T + C_T) \leq \varphi(\bar{W}, \bar{C}_T)\}, \tag{8}$$

which can be interpreted as the minimum money W one needs to charge so that his or her risk after selling $\varphi(\bar{W} + W, \bar{C}_T + C_T)$ is not worse than the risk before selling $\varphi(\bar{W}, \bar{C}_T)$.

Similarly, the indifference price for buying the claim C_T can be defined as follows:

$$\pi_b(\bar{W}, \bar{C}_T, C_T) = \sup\{W | \varphi(\bar{W} - W, \bar{C}_T - C_T) \leq \varphi(\bar{W}, \bar{C}_T)\}, \tag{9}$$

which can be seen as the maximum money one can pay so that his or her risk after buying and receiving the claim C_T does not worsen afterwards.

As shown by Pennanen (2012, 2014), $\varphi(W, C_T)$ is convex, non-increasing in W and non-decreasing in C_T . This is not surprising, as increasing an initial wealth should not worsen an agent’s optimal risk, and an agent with higher liability should have a less favorable position. In practice, given a future liability C_T , to compute the indifference pricing for selling (buying), the bisection method can be employed to find the minimum (maximum) initial wealth W which satisfies the inequality condition.

Table 3 The payoffs of the tradable assets

Assets	Payoffs of tradable assets
Cash	e^{rT}
Call option	$\max\{S_T - K, 0\}$
Put option	$\max\{K - S_T, 0\}$

Unlike traditional dynamic hedging that requires continuous trading of the underlying asset, we employ an option-hedge-option strategy where exotic derivatives are hedged using a static portfolio of standard options. The buy-and-hold hedging strategy is the difference between the two portfolios, $\mathbf{x} - \bar{\mathbf{x}}$, where \mathbf{x} is the optimized portfolio after the claim C_T is included in the optimization and $\bar{\mathbf{x}}$ is the optimized portfolio in the case where liability is absent.

3 The Market and Probabilistic Point of Views

To illustrate the technique, standard put and call options in the S&P 500 market were used as tradable assets. In other words, we want to find the combination of these standard put and call options such that CVaR is minimized. In addition, in the pricing section of the exotic derivatives, these standard put and call options were used to determine the hedging strategy.

In this paper, to illustrate the technique, we use 199 standard options available for trading on the S&P 500 Mini index on May 19, 2020, at 12:00:00. The strikes are from 250 to 350, with a maturity of one month ($T = 1/12$). Tables 1 and 2 provide examples of snapshots for the call and put options, respectively. The underlying price at this specific time was 295.42 USD. The quotes, obtained from Bloomberg, include both bid and ask prices and sizes. A bid price is the price one will get if he or she wants to sell the option, whereas an ask price is the price one will need to pay in order to buy. Bid and ask sizes are the maximum amounts that one can sell and buy, respectively. The lot size is 100 units per contract.

Despite the availability of 199 standard options on the S&P 500 Mini index, there is a difference between the bid and ask prices for the options, known as the bid-ask spread. This spread requires a strategic approach to options trading that involves both selling and buying. To address this, we duplicate the 199 options, resulting in a total of 398 options. The first 199 options are restricted to selling, while the remaining 199 options can only be bought. To manage this, we introduce a portfolio

Table 4 The current index, the compound interest rate (r), the base-case parameters of the VG distribution and the required return (Q)

Index	r	μ	σ	ν	θ	Q
295.42	0	0.000001	0.2	0.01	0	5%

vector \mathbf{x} , which includes cash holdings x^0 , selling weights x^b , and buying weights x^a . Therefore, $\mathbf{x} = [x^0 \ x^b \ x^a]$. The associated cost vector is denoted by $S_0 = [S_0^0 \ S_0^b \ S_0^a]$.

Theorem 1 The probability density function (PDF) for an underlying price S_T at maturity time T following a variance gamma process is defined as Madan et al. (1998) and Madan and Seneta (1990):

$$f_{VG}(S_T) = \frac{2 \exp(\theta y / \sigma^2)}{\sqrt{(T/\nu)} 2\pi \sigma \Gamma\left(\frac{T}{\nu}\right)} \left(\frac{y^2}{2\sigma^2/\nu + \theta^2}\right)^{\frac{T}{2\nu} - \frac{1}{4}} B_{\left(\frac{T}{\nu} - \frac{1}{2}\right)} \frac{1}{\sigma^2} \sqrt{y^2(2\sigma^2/\nu + \theta^2)}, \tag{10}$$

where $B_\eta(\cdot)$ is the modified Bessel function of the second kind of order η , $\Gamma(\cdot)$ is the gamma function, and

$$y = \log(S_T) - \log\left(S_0 - \mu T - \frac{T}{\nu} \log\left(1 - \theta\nu - \frac{1}{2}\sigma^2\nu\right)\right).$$

Table 3 shows the payoffs at maturity time T of cash, call options, and put options corresponding to a specified strike price K . Following the calculation of these payoffs, they are concatenated and the return vector $\mathbf{R}(S_T)$, as defined in (5), is expressed as the ratio between the payoff vectors and their respective cost vectors. Critically, we incorporate market prices by using ask prices for long positions and bid prices for short positions as shown in Table 1 and 2, thus accounting for the actual cost of establishing these positions in the market. In this work, the underlying price S_T , which is the S&P 500 mini index, is modeled by the Variance Gamma (VG) process, which is obtained by evaluating the Geometric Brownian Motion (GBM) with the drift parameter θ at the random time change that follows a gamma process. While we employ the VG distribution for its ability to capture market features like skewness and fat tails, the framework introduced in Sect. 2.1 can accommodate other well-defined probability distributions such as normal, log-normal, and t-distribution, with only minor variations in computation time. These parameters (μ, σ, ν, θ) are calibrated using Maximum Likelihood

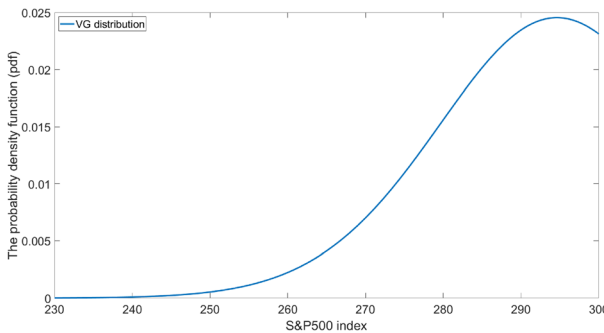


Fig. 1 The VG-PDF of stock prices from 230 to 300

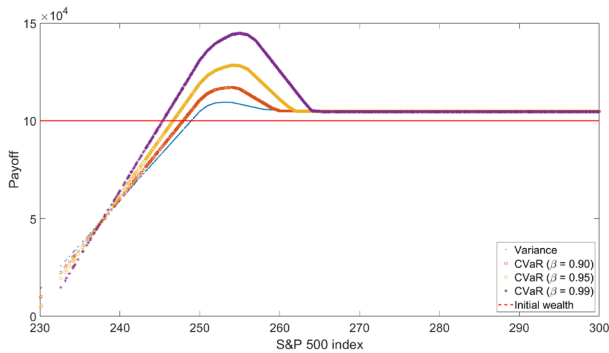


Fig. 2 The payoffs of portfolios for each models

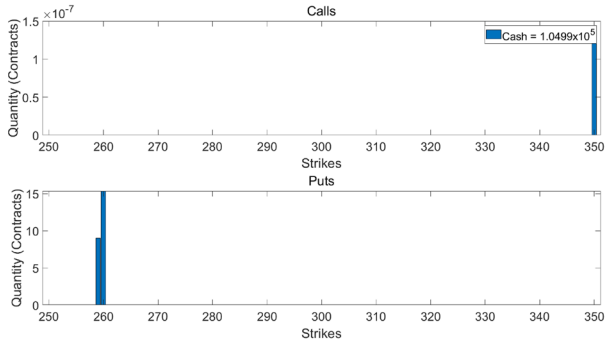
Estimation (MLE) following Madan et al. (1998) and Madan and Seneta (1990). The parameters estimate satisfy,

$$L(\mu, \sigma, \nu, \theta) = \sum_i \log(f_{VG}(S_T^i)),$$

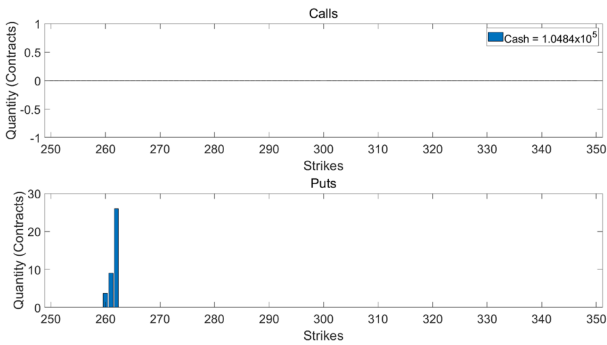
where $f_{VG}(S_T)$ is the VG probability density function defined in Theorem 1. The variance gamma process is able to capture the dynamics of the underlying price better than the original GBM model, which falls under a lognormal assumption, as it has the ability to model both skewness and kurtosis. The kurtosis, asymmetry and volatility are modeled by ν , θ and σ , respectively. This makes more sense, as the distributions of the stock prices evidently have fat tails. The density function of the VG process is known and is given in Theorem 1. Unless otherwise stated, the initial wealth W is 100,000 USD, the required return M is 105,000 USD, and the modeling parameters are as in Table 4.

4 Results

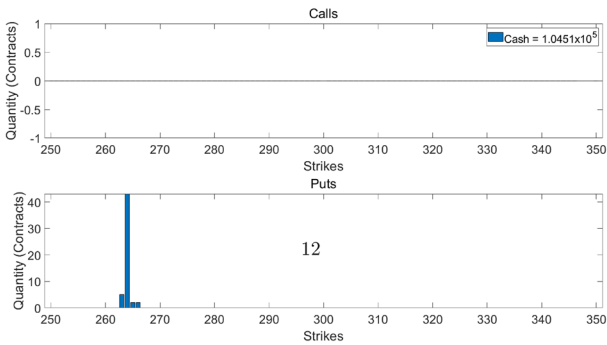
Our analysis consists of three key subsections. The first subsection examines optimized portfolios, analyzing how they vary with modeling parameters and how their payoffs change with the underlying asset price at maturity time T (excluding future liabilities C_T). The second subsection analyzes indifference prices and their corresponding hedging portfolios for exotic options. The third subsection evaluates our portfolio optimization through backtesting, using historical data to assess investment strategy performance and measure hedging accuracy.



(A) $\beta = 0.90$



(B) $\beta = 0.95$

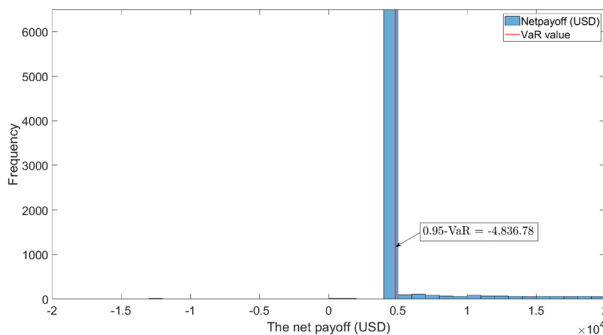


(C) $\beta = 0.99$

Fig. 3 The investment units of call and put options for each confidence levels

Table 5 The expectation (Portmean), the standard deviation (PortSD), the VaR and CVaR of the CVaR minimization portfolio

Model	PortSD	CVaR		
		$\beta = 0.90$	$\beta = 0.95$	$\beta = 0.99$
Variance	1,912.5268	-3,874.5997	-3,108.2169	2,987.5383
CVaR ($\beta = 0.90$)	1,925.7784	-4,320.1462	-3,650.8305	1,703.6950
CVaR ($\beta = 0.95$)	2,579.9318	-4,274.1358	-3,711.4875	789.6989
CVaR ($\beta = 0.99$)	4,265.0500	-4,034.2231	-3,557.2646	258.5949

**Fig. 4** The histogram of net payoff for base-case at the confidence level $\beta = 0.95$

4.1 Portfolio Optimization Without a Future Liability

We begin by assuming no future liability ($C_T = 0$) and seek to determine an optimal portfolio that minimizes CVaR while meeting a capital requirement of 105,000 USD at maturity time T . We set the maturity period to one month ($T = 1/12$) and model the index value using a VG process with base-case parameters from Table 4.

For the simulation of the index values (S_T), Fig. 1 provides a representation of the probability density function within the index range of 230 to 300. The function reaches its maximum probability density, approximately 0.025, at an index value of 295, which is close to the spot price of 295.42.

Figures 2 and 3 illustrate the payoff as functions of the index value at expiration across various confidence levels corresponding to the optimized portfolios. Table 5 summarizes the minimum variance and CVaR values, which were obtained through 100,000 out-of-sample of the mean-variance optimization and 500 points of the Gauss-Legendre quadrature rule of portfolio optimization with CVaR. The optimization process was conducted using MATLAB version R2023b, with an average computation time of 3.0291 s on a Windows 11 Pro system, which is equipped with a 12th Gen Intel(R) Core(TM) i7-12700 2.10 GHz processor and 16.0 GB of RAM.

Figure 3 illustrates the optimal portfolios \mathbf{x} for various confidence levels. As the confidence level increases, the cash holdings decrease. This trend occurs because higher confidence levels indicate greater certainty about expected returns and risks, reducing

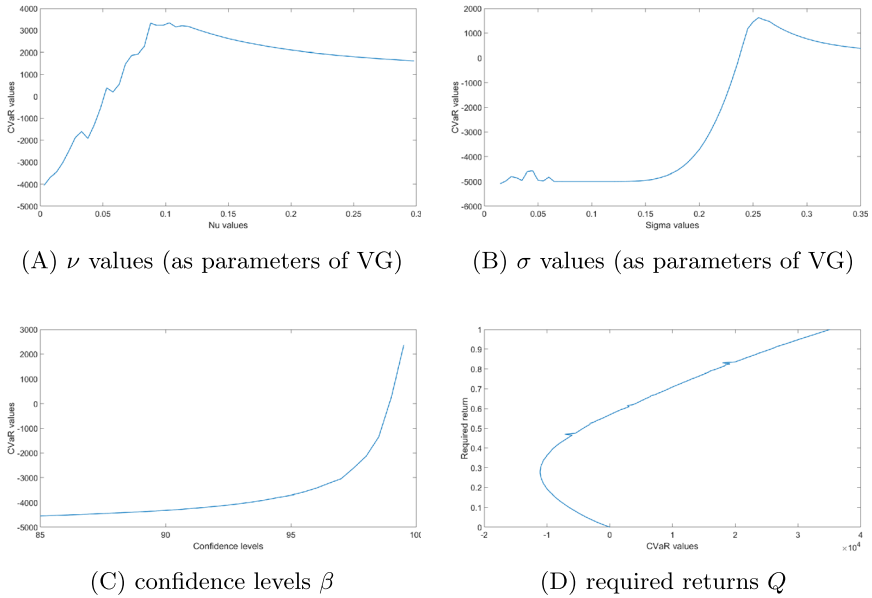


Fig. 5 The minimum CVaR of the portfolio by changing any parameters

the need for cash as a buffer. With increased confidence, investors are more inclined to allocate capital to riskier assets, such as put and call options, rather than holding cash. Consequently, the investment strategy becomes more aggressive, with a larger proportion of the portfolio invested in assets with higher expected returns, particularly in options. Across all confidence levels, the strategy consistently includes purchasing put options with strike prices between 260 and 270. However, at the confidence level $\beta = 0.90$, the strategy also features a minor purchase of a call option with a strike price of $K = 350$.

Table 5 presents the expected value of the portfolio payoffs (Portmean), along with the portfolio standard deviations (PortSD) and CVaR values for each confidence level. When focusing on minimizing the variance of the portfolio payoff, the PortSD is the lowest at 1,912.53 USD, compared to other models. When optimizing the CVaR of the portfolio for each confidence level, the CVaR value is minimized at the specific confidence level. As the confidence level increases, the CVaR values of the optimal portfolios also increase. For example, the $\beta = 0.95$ CVaR model, the PortSD is 2,579.93 USD, while the CVaR values for confidence levels $\beta = 0.90, 0.95$ and 0.99 are $-4,274.14, -3,711.49$ and 789.70 , respectively. This demonstrates that losses occur at $\beta = 0.99$ due to the positive CVaR value.

Figure 4 illustrates the histogram of net payoffs for the optimized portfolio, assessed at the confidence level $\beta = 0.95$ using 100,000 out-of-sample simulated prices. The maximum net payoff at the confidence level of 0.95, or the 0.95-VaR, is $-4,836.78$ USD. Due to the negative VaR value, this means that, with the confidence level $\beta = 0.95$, the portfolio will not lose money, but instead will earn at least 4,836.78 USD, and there is a 5% chance that the gain could be smaller than this amount or turn

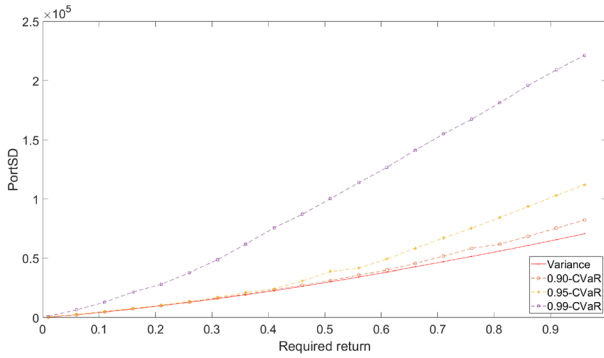


Fig. 6 The relations between the required returns and the portSD values of the minimum SD and CVaR models

into a loss. In particular, the distribution extends to the left tail beyond the VaR threshold. Furthermore, as detailed in Table 5, the average net payoff in the worst 5% cases, corresponding to the 0.95-CVaR, is $-3,711.49$ USD.

4.1.1 Changing Parameters

Our evaluation reveals four critical parameters that directly affect CVaR values: the shape parameter (ν), volatility (σ), confidence level (β), and required return (Q). Figure 5 illustrates these relationships through four subfigures:

Figure 5A demonstrates that as ν increases, the CVaR values change from negative (representing profits) to positive (representing losses). This shift occurs because a higher ν introduces greater skewness and heavier tails in the net payoff distribution, leading to increased potential losses. The maximum loss of 3,000 USD, observed at $\nu = 0.1$, illustrates the increased risk associated with higher values of ν .

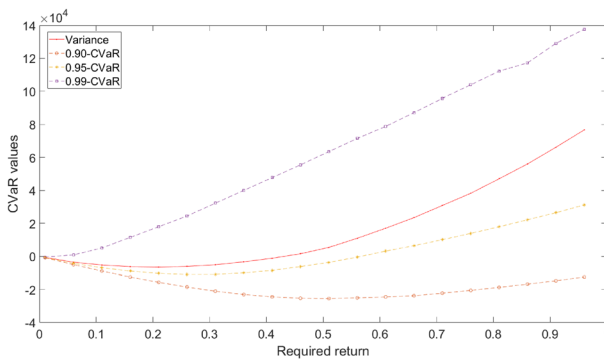


Fig. 7 The relations between the required returns and the CVaR values of the minimum SD and CVaR models

Table 6 The claims for indifference prices of the exotic options

Exotic options	Claims (C_T) for pricing
European call option	$\max\{S_T - K, 0\}$
Quadratic option	$ S_T - K ^2$
Log option	$1000 \log\left(\frac{K}{S_T}\right)$
Digital option	$1000(S_T \geq K)$
Sine option	$1000 \sin\left(\frac{2\pi}{10} S_T\right)$
Butterfly spread option	$100((S_T - 295) - 2(S_T - K) + (S_T - 305))$

Figure 5B illustrates that the volatility has a significant impact on CVaR. Initially, as volatility increases, CVaR gradually increases, indicating that higher volatility generally leads to greater potential losses. The sharp increase in CVaR is observed when volatility reaches $\sigma \geq 0.15$, suggesting that moderate volatility levels considerably increase the risk of extreme losses. However, at higher volatility levels ($\sigma \geq 0.25$), CVaR begins to decrease. This counterintuitive result implies that very high volatility spreads the distribution of returns more widely, which can lead to a reduction in the average of the worst-case losses.

Figure 5C demonstrates an exponential relationship between CVaR values and β . The increase in CVaR is modest for β values between 0.85 and 0.95, but there is a substantial increase at higher confidence levels. This observation aligns with the interpretation of β , as higher values place greater emphasis on extreme downside risks. As β increases, the model accounts for more severe losses in the tail of the distribution, leading to higher CVaR values.

Figure 5D illustrates a logarithmic relationship between CVaR values and Q . As the required return increases, CVaR increases, indicating that higher return targets lead to greater potential losses. However, the rate of increase decreases as the required return becomes more ambitious. This pattern highlights the fundamental trade-off between the pursuit of higher returns and the associated risk of higher losses. Although the potential for greater losses slows at higher levels, the risk remains significant.

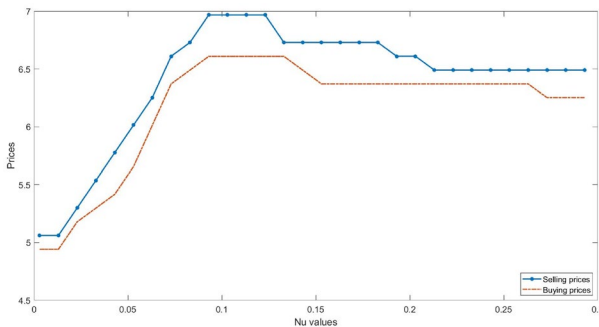


Fig. 8 Selling and Buying prices of the European call option after changing ν values

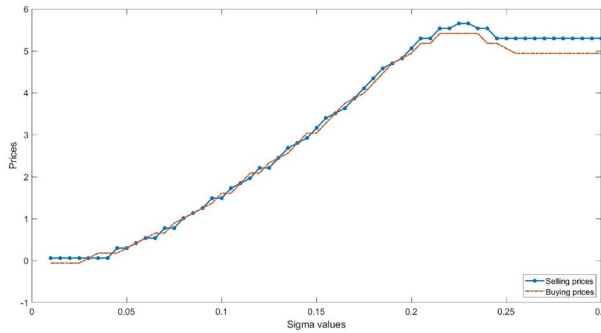


Fig. 9 Selling and Buying prices of the European call option after changing σ values

4.1.2 Compare Results from CVaR Model with the Variance Model

According to Table 5, with a required return of 5%, the portfolio standard deviation (PortSD) of the optimized portfolio from the variance model is lower compared to the CVaR model. Figure 6 illustrates the relationship between PortSD and the required returns for both the variance model and the CVaR model. The analysis reveals that, for any given required return, the variance model consistently yields lower PortSD values than the CVaR model.

This consequence is expected because mean-variance optimization focuses on minimizing overall volatility, resulting in portfolios with lower PortSD. In contrast, CVaR optimization aims to manage extreme losses, which can lead to higher PortSD as it can accept greater overall volatility to reduce the risk of severe losses. Furthermore, within the CVaR framework, PortSD values increase as confidence levels increase, reflecting a higher tolerance for volatility to address potential extreme losses.

Figure 7 illustrates that CVaR optimization generally yields lower CVaR values than the mean-variance model across all required return levels. This indicates that, at various required return levels, the CVaR model typically manages extreme losses more effectively, resulting in lower CVaR values compared to the variance model.

Table 7 The selling prices of the portfolio optimization with CVaR

Claims	Selling Prices			
	Variance	0.90-CVaR	0.95-CVaR	0.99-CVaR
European Call option	5.5611	4.8804	5.0640	6.2823
Quadratic option	544.7616	343.4563	390.2030	732.5172
Log option	38.6350	15.7619	16.8800	24.7407
Digital option	513.3850	426.4021	462.3008	688.9415
Sine option	351.8753	200.7794	339.0002	744.5168
Butterfly spread option	68.6890	65.3791	73.4234	126.2617

Table 8 The buying prices of the portfolio optimization with CVaR

Claims	Buying Prices			
	Variance	0.90-CVaR	0.95-CVaR	0.99-CVaR
European Call option	5.5580	4.8327	4.9829	5.9843
Quadratic option	540.5407	339.5033	383.7786	687.6421
Log option	38.6086	15.6808	16.6655	23.5248
Digital option	462.4805	363.8864	372.1142	460.2337
Sine option	-376.0223	-221.2572	-387.8665	-679.7957
Butterfly spread option	54.7962	46.3724	41.0819	12.9771

As the required return increases, CVaR values in both models tend to increase due to the higher risk associated with achieving greater returns. However, the CVaR model consistently shows lower CVaR values than the variance model, highlighting its superior performance in mitigating extreme losses while accommodating higher return requirements. The exception is observed at the highest confidence level ($\beta = 0.99$), where the CVaR value of the CVaR model may exceed that of the variance model. This suggests that, at very high confidence levels, the CVaR model might indicate higher potential losses due to its focus on extreme tail risks.

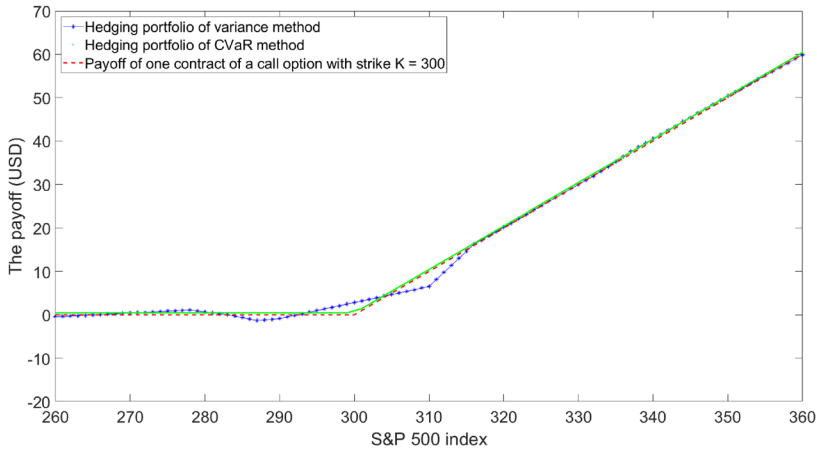
4.2 Portfolio Optimization with a Future Liability

This section investigates minimizing a portfolio in the presence of liabilities. The formal definitions for the price claims of six exotic options (including a European call, a quadratic, a log, a digital, a sine and a butterfly spread option) are defined in Table 6. The simulated prices at maturity (S_T) are generated based on the VG distribution, with a fixed strike price of $K = 300$ USD.

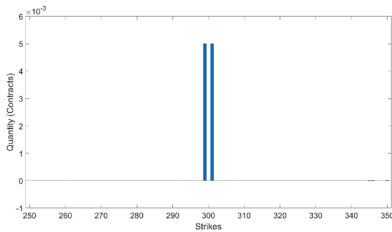
Given that the CVaR value is influenced by VG parameters, such as the shape parameter (ν) and volatility (σ), it is essential to investigate their effects on selling and buying prices. These parameters significantly impact CVaR, thereby affecting the assessment and pricing of financial instruments.

The shape parameter (ν) of the VG process significantly impacts the option price. Figure 8 shows that indifference prices rise sharply for $\nu \in [0, 0.1]$ due to the increased risk of heavier tails, then decrease slightly as ν increases further. Selling prices are higher than buying prices because sellers demand a premium for the risk they assume, influenced by factors such as the bid-ask spread and risk aversion.

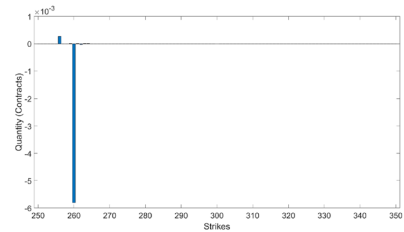
Figure 9 illustrates that both selling and buying prices exhibit variability and an upward trend as volatility increases. In the Variance Gamma (VG) process, higher volatility is associated with elevated option prices due to greater uncertainty and the potential for extreme price fluctuations. The theoretical results of Armstrong et al. in Theorem 5.1 reveal that, in the absence of binding constraints, selling prices are generally higher than buying prices due to risk aversion. Direct observation shows that selling prices exceed buying prices when σ surpasses 0.2. The widening gap between selling and buying prices as volatility increases reflects heightened market



(A)



(B)

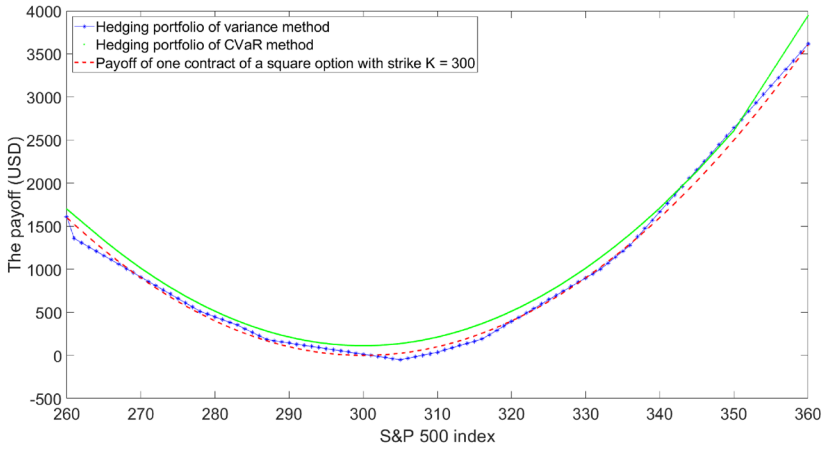


(C)

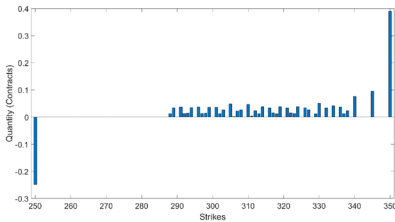
Fig. 10 The payoff of hedging portfolio with the payoff of claim prices of the European call option (A), the hedging portfolios for the call option (B), and the put option (C)

uncertainty, where sellers demand higher premiums to compensate for greater risk, while buyers become more price-sensitive. Although 0.2 is not a theoretically derived critical value but rather an empirical observation based on the specific dataset and model parameters, it provides insight into how volatility influences the relationship between buying and selling prices under the given market conditions.

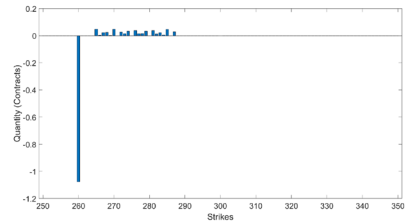
Table 7 illustrates that the CVaR model generally yields lower selling prices for exotic options compared to the variance model. However, this trend reverses at the highest confidence level ($\beta = 0.99$). For instance, at $\beta = 0.95$, the selling and buying prices associated with CVaR claims (as detailed in Tables 7 and 8) are lower than those derived from the variance model, with the exception of the selling prices for a butterfly spread claim. As confidence levels increase, the corresponding increase in CVaR values reflects greater risk exposure. To mitigate this heightened risk, additional initial capital is required, leading to higher selling prices. Thus, the increase in selling prices is attributed to the need for additional wealth to manage the elevated risk associated with higher CVaR values.



(A)



(B)

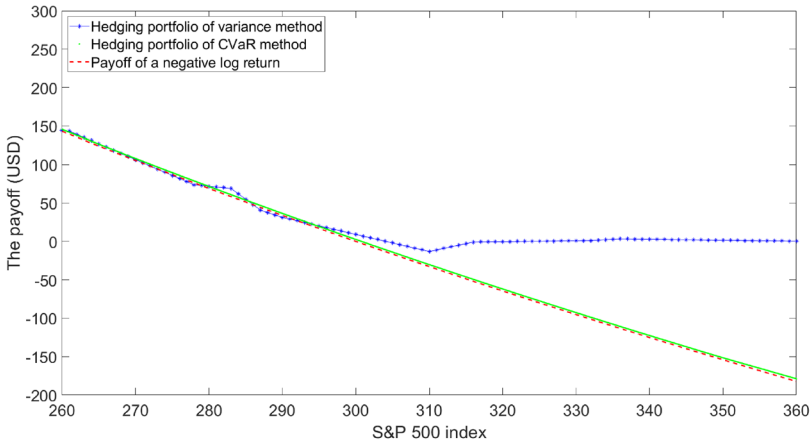


(C)

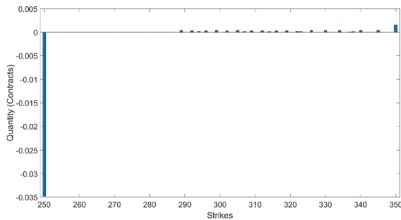
Fig. 11 The payoff of hedging portfolio with the payoff of claim prices of a quadratic option (A), the hedging portfolios for the call option (B) and the put option (C)

The bid and ask prices for a European call option with a strike price $K = 300$ USD are 5.5 and 5.65, as presented in Table 1. In contrast, the CVaR-based selling and buying prices at a confidence level $\beta = 0.95$ are 5.0640 and 4.9829, respectively, which are lower than market prices. This difference suggests that the CVaR model's prices are lower due to different risk assessments or because the model does not account for transaction costs and market liquidity. Table 8 also shows that selling prices typically exceed buying prices for exotic options, with the exception of the sine option.

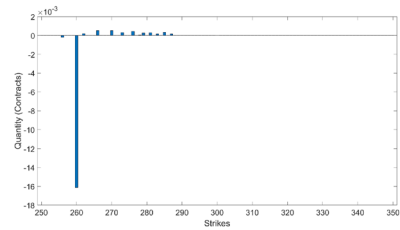
Figures 10-15 illustrate the payoffs of hedging portfolio for the variance and CVaR methods and the hedging portfolios for call and put options with any exotic options. Each claim is represented by three subfigures: a graph comparing the payoffs of the hedging portfolio and the option claim, as well as bar graphs illustrating the composition of hedging portfolios for both call and put options. The results demonstrate that the payoffs of the hedging portfolio generated through the CVaR optimization method (depicted by the green graph) consistently exceed those of the option claim (represented by the red graph) across all claims, with both showing



(A)



(B)



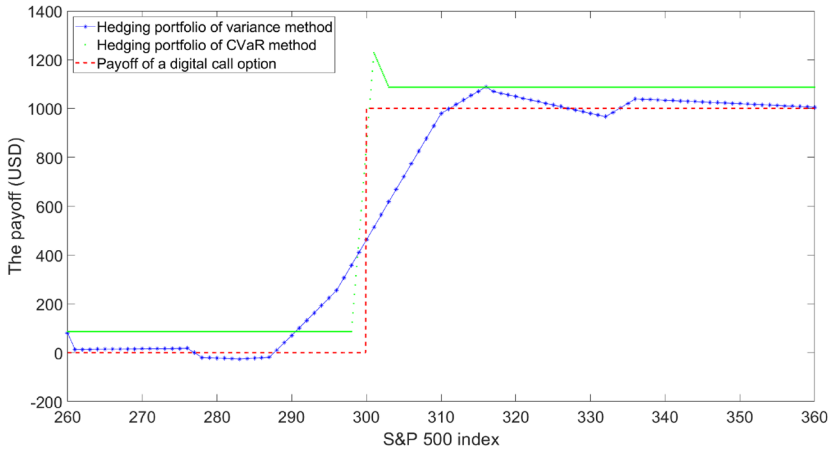
(C)

Fig. 12 The payoff of hedging portfolio with the payoff of claim prices of a log option (A), the hedging portfolios for the call option (B) and the put option (C)

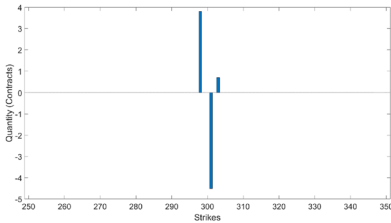
similar trends. This indicates a generally favorable outcome, suggesting that the hedge effectively reduces risk while maintaining a desirable payoff profile. For instance, in cases where the option claim exhibits significant downside risk, the hedging portfolio mitigates this risk, demonstrating the effectiveness of the hedging strategy.

Furthermore, Fig. 10A compares the hedging portfolios for a call option with a strike price of $K = 300$ USD, using both the variance method (blue graph) and the CVaR method (green graph). The red line represents the payoff of a single call option contract, which becomes positive when the stock price (S_T) exceeds the strike price. In most cases, the payoffs from hedging portfolios generated by the variance method are lower than those obtained using the CVaR method, as shown in Figs. 10A, 11, 12, 13, 14 and 15A. However, a notable exception is observed with the log option, where the variance method consistently results in higher payoffs compared to the CVaR method.

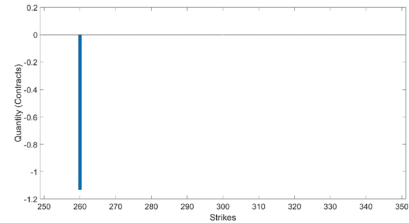
In addition, the bar graphs illustrate the composition of the hedging portfolio for both call and put options, showing the number of contracts (100 units each)



(A)



(B)



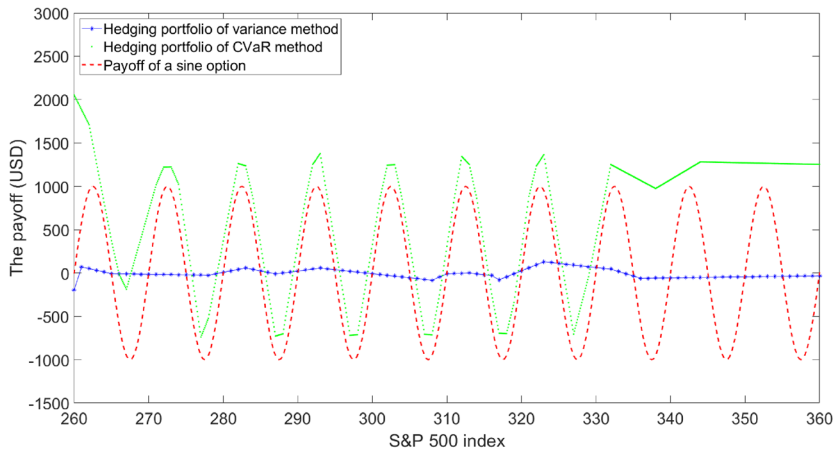
(C)

Fig. 13 The payoff of hedging portfolio with the payoff of claim prices of a digital option (A), the hedging portfolios for the call option (B) and the put option (C)

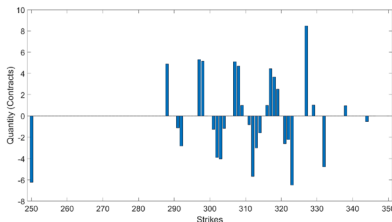
involved. A positive contract quantity indicates a recommendation to purchase the option at the specified strike price, whereas a negative quantity suggests selling the option. For instance, Figs. 10B and 10C display the optimal number of contracts for the call and put options. The hedging strategy recommends purchasing 2 call options at a strike price of approximately 300, buying a put option at a strike price around 255, and selling a put option at a strike price of approximately 260.

4.3 Dynamic Portfolio Optimization

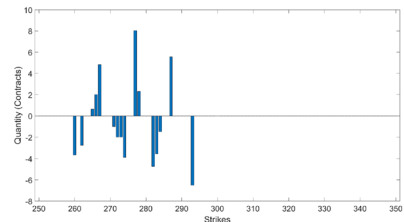
To investigate the performance of the scheme in practice, we extended the strategy and applied it in a dynamic framework. Assuming that an agent begins an investment at time $t = 0$, he or she would have an optimal portfolio x_0 to hold until maturity $t = T_1$. At $t = T_1$, the options portfolio will expire and the agent's net wealth, W_{T_1} will all be in cash. The agent then reinvests W_{T_1} again in options that expire at T_2 . Performing this process from maturities $T_1, T_2, T_3, \dots, T_n$ using historical data, we



(A)



(B)



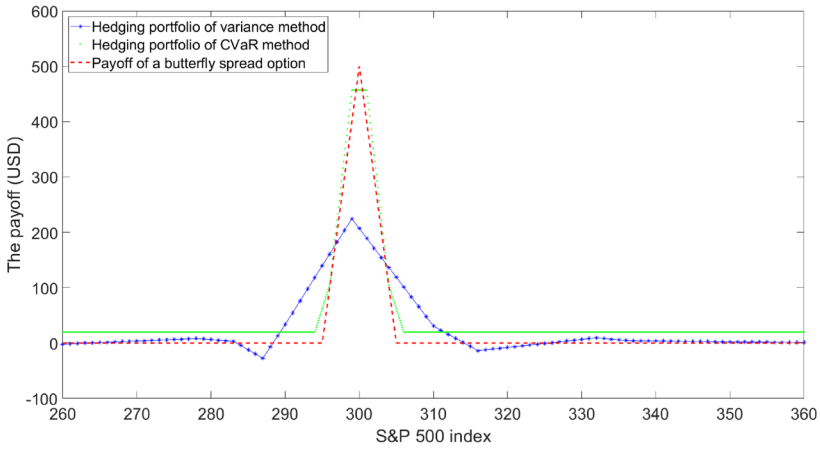
(C)

Fig. 14 The payoff of hedging portfolio with the payoff of claim prices of a sine option (A), the hedging portfolios for the call option (B) and the put option (C)

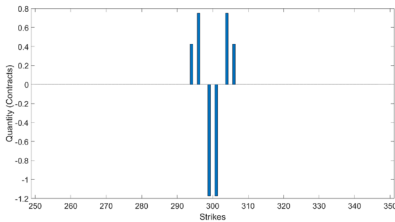
can investigate the profits and losses of the technique over time under this dynamic trading strategy. This approach enables us to compute the portfolio's payoff and its precise percentage return. Additionally, we consider the error between the payoff of the hedging portfolio and that of a European call option at the option's expiration date.

As shown in Table 9, in the first row, we invested in options on March 27, 2020 and held the portfolio until the expiration date on April 17, 2020. The investment period was 21 days, resulting in a payoff of approximately 104,884 USD and an exact return of 4.89%. This return is close to the target return of 5%. Figure 16 illustrates that the exact returns of our optimized portfolios, held from the investment date to expiration, are generally distributed around the 5% target return.

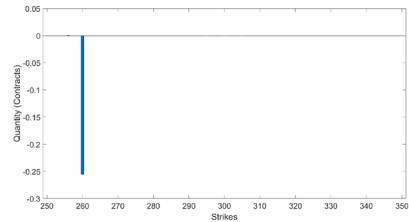
Table 10 demonstrates that an investment in the optimal portfolio, initiated on March 27, 2020 and held until April 17, 2020, with hedging based on indifference pricing, resulted in a minor error of approximately 8.22. Most of these errors are negligible, indicating that the hedging portfolio's payoff closely aligns with the European call option's payoff. In addition, the selling prices are influenced by the



(A)



(B)



(C)

Fig. 15 The payoff of hedging portfolio with the payoff of claim prices of a butterfly option (A), the hedging portfolios for the call option (B) and the put option (C)

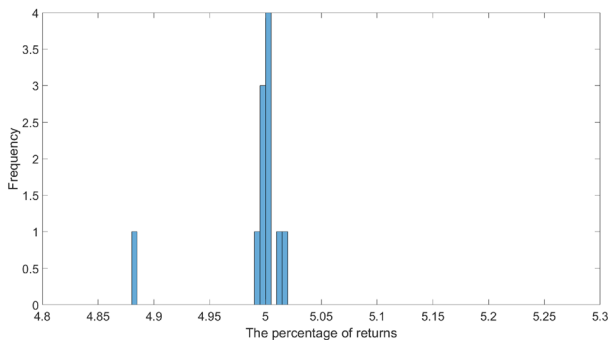
opening prices due to underlying market conditions, which contribute to higher indifference prices. Figure 17 shows the distribution of hedging errors, with a mean error of zero, suggesting that the hedging strategy is generally accurate.

5 Conclusion and Discussion

This paper has explored portfolio optimization through the CVaR model and compared it with the traditional mean-variance approach. Unlike traditional dynamic hedging that requires continuous trading of the underlying asset, we employ an option-hedge-option strategy where exotic derivatives are hedged using a static portfolio of standard options. The buy-and-hold hedging strategy represents the difference between two portfolios: \mathbf{x} and $\bar{\mathbf{x}}$. Here, \mathbf{x} is the optimized portfolio including claim C_T , while $\bar{\mathbf{x}}$ is the optimized portfolio without liabilities. Our investigation is divided into three distinct parts.

Table 9 The trading date, the expiration date, the time to maturity (TTM), the opening price, the payoff of this portfolio (Payoff) and the precise percentage return for each month excluding claims

Trading date	Expiration date	TTM (days)	Opening Price	Payoff	Return
27.03.2020	17.04.2020	21	285.38	104,883.6004	4.8836
24.04.2020	15.05.2020	21	282.37	105,000.0000	5.0000
28.05.2020	19.06.2020	22	314.17	105,000.0007	5.0000
26.06.2020	17.07.2020	21	321.88	105,000.0000	5.0000
31.07.2020	21.08.2020	21	337.92	105,000.5183	5.0005
28.08.2020	18.09.2020	21	335.37	104,992.3657	4.9924
25.09.2020	16.10.2020	21	348.96	104,999.9982	5.0000
30.10.2020	20.11.2020	21	357.50	104,999.0871	4.9991
25.11.2020	18.12.2020	23	370.97	105,001.3615	5.0014
23.12.2020	15.01.2021	23	376.72	105,012.0535	5.0121
28.01.2021	19.02.2021	22	392.07	105,018.1711	5.0182
17.03.2021	16.04.2021	30	417.25	105,887.0086	5.8870
30.04.2021	21.05.2021	21	416.87	111,507.5499	11.5075
27.05.2021	18.06.2021	20	417.09	118,532.1605	18.5322
23.06.2021	23.07.2021	30	437.52	99,708.5684	-0.2914
29.07.2021	20.08.2021	22	440.23	157,417.5880	57.4176

**Fig. 16** The histogram of the exact percentage of returns for an optimal portfolio excluding claims

First, we use linear programming to model portfolio optimization, aiming to minimize CVaR while considering the percentage return and the optimal allocation among cash, call options, and put options. Our results indicate that the CVaR model is particularly sensitive to the VG distribution parameters, including the shape parameter (ν), volatility (σ), confidence level and required return. This sensitivity reveals a nuanced trade-off between risk and return, where the CVaR model often achieves lower risk levels even with higher standard deviations compared to the variance model.

Second, we extend the CVaR-based portfolio optimization to include future claims. There are six exotic options such as European call, quadratic, log, digital,

Table 10 The trading date, the expiration date, the time to maturity (TTM), the opening price, the selling price and the hedging error with a european call option as a claim

Trading date	Expiration date	TTM (days)	Opening Price	Selling Price	Hedging error
27.03.2020	17.04.2020	21	285.38	0.0596	8.2272
24.04.2020	15.05.2020	21	282.37	0.6337	-0.0020
28.05.2020	19.06.2020	22	314.17	8.5297	0.0060
26.06.2020	17.07.2020	21	321.88	7.1838	-0.0036
31.07.2020	21.08.2020	21	337.92	23.8597	0.1717
28.08.2020	18.09.2020	21	335.37	46.8731	-3.2677
25.09.2020	16.10.2020	21	348.96	25.5406	0.0012
30.10.2020	20.11.2020	21	357.50	26.5956	-17.5859
25.11.2020	18.12.2020	23	370.97	60.0553	0.0902
23.12.2020	15.01.2021	23	376.72	67.1685	-0.0013
28.01.2021	19.02.2021	22	392.07	76.7350	-0.0076
17.03.2021	16.04.2021	30	417.25	90.5573	-0.0250
30.04.2021	21.05.2021	21	416.87	114.2323	0.0552
27.05.2021	18.06.2021	20	417.09	115.3767	0.0012
23.06.2021	23.07.2021	30	437.52	125.0148	-0.0021
29.07.2021	20.08.2021	22	440.23	137.4245	2.2016

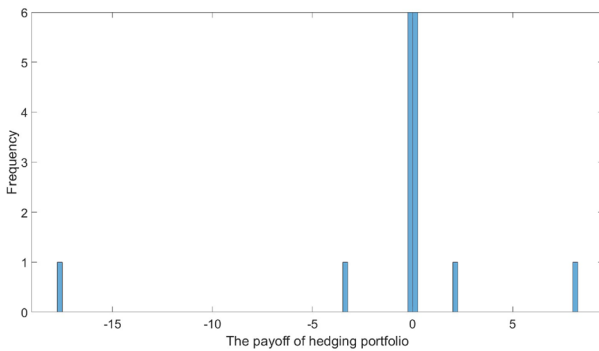


Fig. 17 The histogram of the hedging errors

sine and butterfly spread options. The extended CVaR model generally outperforms the variance model in managing risk, especially for exotic options. The exception was the log option, where the variance model showed better performance. This suggests that the CVaR model is effective in optimizing hedging strategies, particularly for complex option types.

Finally, we consider the backtesting part. We evaluated CVaR-based portfolio optimization by applying it to historical data, taking both long and short positions. We found that the precise rate of returns from the portfolios optimized without liabilities are closed to the required return percentages. Furthermore, the payoff from the hedging portfolio was found to be equivalent to the European call option claim,

with hedging errors approaching zero. Our framework demonstrates that static option-based hedging, combined with CVaR optimization, can effectively manage complex derivative portfolios while reducing transaction costs compared to traditional dynamic strategies.

This research highlights the differential impacts of risk measures and pricing methods on portfolio optimization. The CVaR model emerges as a robust framework, particularly for portfolios involving exotic options. While our strategy can be applied in complete markets, it is unnecessary since perfect replication exists, and it would naturally converge to standard Black-Scholes hedging as trading frictions disappear. Its true value lies in incomplete markets, where replication is infeasible due to trading constraints and transaction costs. Future research should explore its application in diverse financial markets and economic conditions to further validate its effectiveness and practical utility. Expanding the analysis to include other financial instruments, such as derivatives and credit risk, and incorporating behavioral factors could provide a more comprehensive understanding of the effectiveness of the CVaR model in real-world applications. Implementing these suggestions could further enhance the model's robustness and applicability in various financial contexts.

Funding This project is funded by National Research Council of Thailand (NRCT): Contract number N42A660925 (2023).

Declarations

Conflict of interests The authors have not disclosed any competing interests.

Open Access This article is licensed under a Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 International License, which permits any non-commercial use, sharing, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if you modified the licensed material. You do not have permission under this licence to share adapted material derived from this article or parts of it. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by-nc-nd/4.0/>.

References

- Abdul Razak, H. N., Maasar, M., Hafidzuddin, N. H., Chun Lee, E. S., et al. (2019). Portfolio optimization of risky assets using mean-variance and mean-cvar. *Journal of Academia*, 7(1), 25–32.
- Armstrong, J., Pennanen, T., & Rakwongwan, U. (2018). Pricing index options by static hedging under finite liquidity. *International Journal of Theoretical and Applied Finance*, 21(06), 1850044.
- Artzner, P. (1997). Thinking coherently. *Risk*, 68–71.
- Artzner, P., Delbaen, F., Eber, J.-M., & Heath, D. (1999). Coherent measures of risk. *Mathematical Finance*, 9(3), 203–228.
- Black, F., & Scholes, M. (1973). The pricing of options and corporate liabilities. *Journal of Political Economy*, 81(3), 637–654.
- Carmona, R. (Ed.). (2008). *Indifference Pricing: Theory and Applications*. New Jersey: Princeton University Press.
- Carr, P., Wu, L. (2002). Static hedging of standard options. CRIF Working Paper series.

- Derman, E., Ergener, D., & Kani, I. (2000). Static options replication. *Journal of Derivatives* 2(4).
- Gaivoronski, A. A., & Pflug, G. (2005). Value-at-risk in portfolio optimization: properties and computational approach. *Journal of Risk*, 7(2), 1–31.
- Guo, X., Chan, R. H., Wong, W.-K., & Zhu, L. (2019). Mean-variance, mean-var, and mean-cvar models for portfolio selection with background risk. *Risk Management*, 21, 73–98.
- Guo, G., & Oblój, J. (2019). Computational methods for martingale optimal transport problems. *The Annals of Applied Probability*, 29(6), 3311–3347.
- Hodges, S. (1989). Optimal replication of contingent claims under transaction costs. *Review Futures Market*, 8, 222–239.
- Hull, J. C. (2019). *Options, futures and other derivatives*. Pearson.
- Ilhan, A., Jonsson, M., & Sircar, R. (2004). Portfolio optimization with derivatives and indifference pricing. *Indifference Pricing* (ed. Carmona), 181–210.
- Ilhan, A., Jonsson, M., & Sircar, R. (2009). Optimal static-dynamic hedges for exotic options under convex risk measures. *Stochastic Processes and their Applications*, 119(10), 3608–3632.
- Kosapong, B., Boonserm, P. & Rakwongwan, U.: (2022). Options portfolio optimization of exotic options written on mini S & P500 index in an illiquid market with conditional value-at-risk (cvar). *Thai Journal of Mathematics*, 169–183.
- Madan, D. B., & Seneta, E. (1990). The variance gamma (vg) model for share market returns. *Journal of business*, 511–524.
- Madan, D. B., Carr, P. P., & Chang, E. C. (1998). The variance gamma process and option pricing. *Review of Finance*, 2(1), 79–105.
- Markowitz, H. M. (1968). *Portfolio Selection*. USA: Yale University Press.
- Merton, R. C. (1973). Theory of rational option pricing. *The Bell Journal of Economics and Management Science*, 4, 141–183.
- Pennanen, T. (2009). Indifference pricing in illiquid markets. manuscript.
- Pennanen, T., & Koivu, M. (2002). *Integration Quadratures in Discretization of Stochastic Programs*. Mathematisch-Naturwissenschaftliche Fakultät II, Institut für Mathematik, Berlin: Humboldt-Universität zu Berlin.
- Pennanen, T. (2012). Introduction to convex optimization in financial markets. *Mathematical Programming*, 134, 157–186.
- Pennanen, T. (2014). Optimal investment and contingent claim valuation in illiquid markets. *Finance and Stochastics*, 18, 733–754.
- Rockafellar, R. T. (2015). *Convex Analysis*. New Jersey: Princeton University Press.
- Rockafellar, R. T., & Uryasev, S. (2000). Optimization of conditional value-at-risk. *Journal of Risk*, 2, 21–42.
- Rockafellar, R. T., & Uryasev, S. (2002). Conditional value-at-risk for general loss distributions. *Journal of Banking & Finance*, 26(7), 1443–1471.
- Sarykalin, S., Serraino, G., & Uryasev, S. (2008). Value-at-risk vs. conditional value-at-risk in risk management and optimization. In: *State-of-the-art Decision-making Tools in the Information-intensive Age*, pp. 270–294. Informs, USA.
- Strub, M. S., Li, D., Cui, X., & Gao, J. (2019). Discrete-time mean-cvar portfolio selection and time-consistency induced term structure of the cvar. *Journal of Economic Dynamics and Control*, 108, 103751.
- Sullivan, M. A. (2000). Valuing american put options using gaussian quadrature. *The Review of Financial Studies*, 13(1), 75–94.
- Wang, X., Zhu, Y., & Tang, P. (2024). Uncertain mean-cvar model for portfolio selection with transaction cost and investors' preferences. *The North American Journal of Economics and Finance*, 69, 102028.
- Wilmott, P. (2013). *Paul Wilmott on Quantitative Finance*. England: John Wiley & Sons.
- Xu, M. (2006). Risk measure pricing and hedging in incomplete markets. *Annals of Finance*, 2(1), 51–71.
- Yamphram, P., Sutthimat, P., & Rakwongwan, U. (2023). Pricing and hedging index options under mean-variance criteria in incomplete markets. *Computation*, 11(2), 30.